Hida Families over totally real fields

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Hida family over totally real flds.

p = prime

lots of congruences mod <u>powers of p</u> between (normalized) eigenforms $S_k(p^\alpha; \mathbb{Z}_p)$.

(Put these in <u>families</u>)

GL_1 theory: Observation

$n \equiv 1 \pmod p$ $\qquad$ $\mathbb{Z} \ni s \mapsto n^s$ can be extended p-adic analytically (i.e. congruences btw these char's)

$\mathbb{Q}_\infty/\mathbb{Q}$ cyclotomic $\mathbb{Z}_p$-ext'n.

$$\begin{tikzcd}
G_{\mathbb{Q},\{p,\infty\}} \arrow[r, two heads] \arrow[drrr, dashed, "\chi_{cyc}^{k-1}"'] & \Gamma \cong Gal(\mathbb{Q}_\infty/\mathbb{Q}) \arrow[r, hook] & \mathbb{Z}_p[[\overset{1+p\mathbb{Z}_p}{\Gamma}]]^\times \arrow[r, "\simeq"] & \mathbb{Z}[[T]]^\times \arrow[d, "\text{mod } P_k = 1+T-(1+p)^{k-1}"] \\
 & & {[1+p]} \arrow[r, mapsto] & 1+T \quad \mathbb{Z}_p^\times
\end{tikzcd}$$

$P_k = 1+T-(1+p)^{k-1}$

"determinant" of the GL_2 case

Moral: "can glue these cyclotomic char. together"

GL_2 $\qquad$ $f_k = \sum_{n \geq 1} a_n(k) q^n$

want to extend $k \in \mathbb{Z}_{\geq 2} \mapsto a_n(k)$ to a <u>p-adic</u> <u>analytic function</u> of k.

$$\begin{tikzcd}
\sigma_\ell \in G_\mathbb{Q} \arrow[r, dashed] \arrow[dr, "\rho_{f_k}"'] & GL_2(\mathbb{Z}_p[[T]]) \arrow[d, "\text{mod } P_k"] \\
 & GL_2(\mathbb{Z}_p)
\end{tikzcd}$$

$\text{trace}(\sigma_\ell) = a_\ell(k)$.

<u>Assumption</u>: <u>Ordinarity</u>, i.e. $\boxed{a_p \text{ is a } p\text{-adic unit.}}$

(conj of Tate (Mazur) Teitelbaum) proven by ? Stevens

Big Galois Rep (Mazur)

wt 1 specialization (Buzzard-Taylor)

Iwasawa theory ← **Hida family** → p-adic L-functions

$F = \mathbb{Q}$.

Selmer complexes
Parity conj

μ-invariant (stay constant in families)

How to construct this family? Constructions: p-adic modular forms

$$\bigcup_{k,\alpha} S_k(\Gamma_1(\underline{N}p^\alpha); \mathbb{Z}_p)$$

$(N,p)=1$, the tame level.

(need make a geometric construction to add additional structure.)

→ ① Using (Betti) cohomology of Shimura varieties
(Eichler-Shimura, Matsushima(?); Harder)

② Geometric definition $H^0(X_1(Np^r)/\mathbb{Z}_p, \omega^k)$ and use Igusa tower
automorphic line bundle.

I. Modular Curves $(F=\mathbb{Q})$

$\underline{k=2}$ Fix $N \geq 4$

$\alpha \in \mathbb{N}$, $X_1(p^\alpha)$ = modular curve of level $\Gamma_1(Np^\alpha)$.

viewed as inside $\mathrm{End}(H^1)$

Hecke algebra $h_\alpha = \mathbb{Z}_p[T_n, n \in \mathbb{N}_{>0}] \curvearrowright H^1(X_1(p^\alpha), \mathbb{Q}_p/\mathbb{Z}_p) =: M_\alpha$

(advantage:) $X_1(p^{\alpha+1}) \twoheadrightarrow X_1(p^\alpha)$ covering $\rightsquigarrow$ $h_{\alpha+1} \twoheadrightarrow h_\alpha$

$M_\alpha \hookrightarrow M_{\alpha+1}$.

$M_\infty := \varinjlim_\alpha M_\alpha$ $\qquad\qquad$ $h_\infty := \varprojlim_\alpha h_\alpha$

$* =$ Pontryagin dual $\qquad\qquad$ ($\mathbb{Z}_p \longleftrightarrow \mathbb{Q}_p/\mathbb{Z}_p$ coeff.)

$$\left.\begin{array}{c} X_\alpha \\ \downarrow \\ X_0(p^\alpha) \end{array}\right] \text{ étale of gp } (\mathbb{Z}/p^\alpha)^\times \longrightarrow h_\alpha \subset \mathrm{End}(M_\alpha).$$

$\Rightarrow$ M_∞^* is a $\mathbb{Z}_p[\![\varprojlim_\alpha (\mathbb{Z}/p^\alpha)^\times]\!] \simeq \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$ -module.

Note the similarity with classical Iwasawa theory!

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$M_\infty^{\mathrm{ord},*} \subset M_\infty$ the biggest direct $\mathbb{Z}_p$-factor where T_p is invertible.

$$\underset{\substack{\cup\\ T_p}}{h_\infty} = \underbrace{h_\infty^{\mathrm{ord}}}_{\text{unit}} \times \underbrace{h_\infty^{ss}}_{\text{top. nilpotent}}$$

$$\begin{tikzcd}[column sep=small] & X_0(p^{\alpha+1}) \arrow[dl] \arrow[dr] & \\ X_0(p^\alpha) & & X_0(p^\alpha) \end{tikzcd}$$

$$\begin{tikzcd} \Gamma_0(p^{\alpha+1}) = \Gamma_0(p^\alpha) \cap \begin{pmatrix} 1 & \\ & p \end{pmatrix}^{-1} \Gamma_0(p^\alpha) \begin{pmatrix} 1 & \\ & p \end{pmatrix} \backslash h \arrow[r, "\begin{pmatrix} p & \\ & 1 \end{pmatrix}", "\sim"'] \arrow[d] & \begin{pmatrix} p & \\ & 1 \end{pmatrix} \Gamma_0(p^\alpha) \begin{pmatrix} p & \\ & 1 \end{pmatrix}^{-1} \cap \Gamma_0(p^\alpha) \backslash h \arrow[d, "pr_2"] \\ \Gamma_0(p^\alpha) \backslash h & \Gamma_0(p^\alpha) \backslash h \arrow[l, "T_p \text{ operator}"'] \end{tikzcd}$$

Ordinary assumption $\Rightarrow$ $H^1_{\mathrm{ord}}(X_0(p^{\alpha+1})) \xrightarrow{\sim} H^1_{\mathrm{ord}}(X_0(p^\alpha)) \simeq \ldots \simeq H^1_{\mathrm{ord}}(X_0(p))$

$\Rightarrow$ there is no ordinary families for Γ_0 !

and

$\Rightarrow$ Families for Γ_1 $\leadsto$ deforming the central character $\begin{pmatrix} \underset{\chi(\cdot)}{\circ} & \\ & \end{pmatrix}$

<u>Thm</u> (Hida):

① h_∞^{ord} is finite and flat over $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$, also over $\Lambda = \mathbb{Z}_p[[1+p\mathbb{Z}_p]] \simeq \mathbb{Z}_p[[T]]$.

$M_\infty^{\mathrm{ord},*}$ is free of finite type over Λ.

② $\alpha \geq 0$, $P_\alpha = (1+T)^{p^{\alpha-1}} - 1$, then $h_\infty^{\mathrm{ord}} / P_\alpha h_\infty^{\mathrm{ord}} \simeq h_\alpha^{\mathrm{ord}}$ $\quad M_\infty^{\mathrm{ord},*} / P_\alpha M_\infty^{\mathrm{ord}} \simeq M_\alpha^{\mathrm{ord},*}$.

(Can recover the finite level Hecke algebras and H^1's.)

III Hilbert modular case.

F = totally real field, $d = [F:\mathbb{Q}]$. $\quad I = \mathrm{Hom}_{\mathbb{Q}}(F, \bar{\mathbb{Q}})$.

Fix $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$.

B = quaternion algebra over F. (e.g. $B = M_2(F)$)

ramified only at some infinite places $I_B \subset I$

$G = \mathrm{Res}_{\mathbb{Q}} B^\times$ (e.g. $G = GL_2(F)/_{\mathbb{Q}}$)

$K=$ open compact of $\quad G(\mathbb{A}_f) = Gl_2(\mathbb{A}_{F,f})$

$\overset{\parallel}{\prod_v K_v} \qquad K_v \subset Gl_2(\mathcal{O}_{F,v})$

Shimura variety

$$Y_K = G(\mathbb{Q}) \backslash G(\mathbb{A}) / KK_\infty$$ (max. compact of $G(\mathbb{R})$) · center

class gp?

Y_K = Shimura variety of dim $= \#(I \backslash I_B) = r$

quasi proj.

smooth if K is small enough

$K = K_1(N) \cap K_0(p^\alpha)$

Hida varieties

Shimura curve.

Two cases: (I) $r=0$ or $I \equiv d \bmod 2$

(II) $d=r \qquad G = Gl_2(F) \qquad$ (Hilbert modular)

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Level at p: $K_0(p^\alpha) \supset K_1(p^\alpha) \supset K_n(p^\alpha)$ ← to put into families more content here

$\underbrace{\begin{pmatrix} * & * \\ & * \end{pmatrix}}_{\text{no families}}$ $\underbrace{\begin{pmatrix} * & * \\ & 1 \end{pmatrix}}_{\text{central char.}}$ $\underbrace{\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}}_{\text{local twist char}}$

$1 + \delta$ - variable families

d - variables.

δ = Leopoldt

Weight: $w = (w_\tau, \tau \in I; w_0)$ $w_\tau \geq 0$ $w_\tau \equiv w_0 \pmod 2$

$v_\tau = \frac{w_0 - w_\tau}{2}$

$M_\alpha(w) = H^r(Y_1(p^\alpha), \mathcal{L}(w; E/\mathcal{O}))$

↪ local system: $\bigotimes_\tau \mathrm{Sym}^{w_\tau} \otimes \det^{\frac{w_0 - w_\tau}{2}}$

$h_\alpha^{(w)} \leftarrow \mathcal{O}[T(\mathbb{Z}/p^\alpha)/\mathcal{O}_E^\times]$ (image?)

G
$\cup$
B = standard Borel
$\cup$
T

$\Lambda^{n,0} = \mathcal{O}[[T(\mathbb{Z}_p)/\mathcal{O}_F^\times]]$
$\downarrow$

$M_\infty^{(w)} = \varinjlim M_\alpha^{(w)}$ $h_\infty^{(w)} = \varprojlim h_\alpha^{(w)}$

$\cup$ $\|$

$M_\infty^{n,0}(w)$ $h_\infty^{n,0}(w) \times h_\infty'(w)$

$Y_1(p^\alpha) \to Y_0(p^\alpha)$

$T(\mathbb{Z}/p^\alpha)/\mathcal{O}_F^\times \simeq (\mathcal{O}_F/p^\alpha)^2/\mathcal{O}_F^\times$

$\begin{pmatrix} uz & \\ & z \end{pmatrix} \longleftarrow (u, z)$

$T_p^0 = \prod_{v | p} T_v^0 = T_p \cdot p^{\text{some power}}$

(so invertible)

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thm (Independence of wt, Hida)

$h_\infty^{n,0}(w) \xrightarrow{\sim} h_\infty^{n,0}(0) = h_\infty^{n,0}$ as Λ-modules

$M_\infty^{n,0}(w) \xrightarrow{\sim} M_\infty^{n,0}(0) = M_\infty^{n,0}$

where the action of $\Lambda^{n,0}$ on wt w is twisted by

$\Lambda_{n,0} \longrightarrow \Lambda_{n,0}$

$[u, z] \longmapsto u^v z^{w_0} [u, z]$

$M_\infty^{n,0}(w) \overset{\text{def}}{=} \varinjlim_\alpha H^r_{n,0}(Y_n(p^\alpha), \mathcal{L}(w, E/\mathcal{O}))$

$= \varinjlim_\alpha H^r_{n,0}(Y_1(p^\alpha), \mathcal{L}(w, p^{-\alpha}\mathcal{O}/\mathcal{O}))$

$=$ " ($p^{-\alpha}\mathcal{O}/\mathcal{O}$)

$$\leq M_{\omega}^{u,d}.$$