

Hida Families over totally real fields

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2:06 PM

Hida family over totally real f.f.s.

$p = \text{prime}$

lots of congruences mod powers of p between (normalized) eigenforms $S_k(\Gamma^0(N); \mathbb{Z}_p)$.

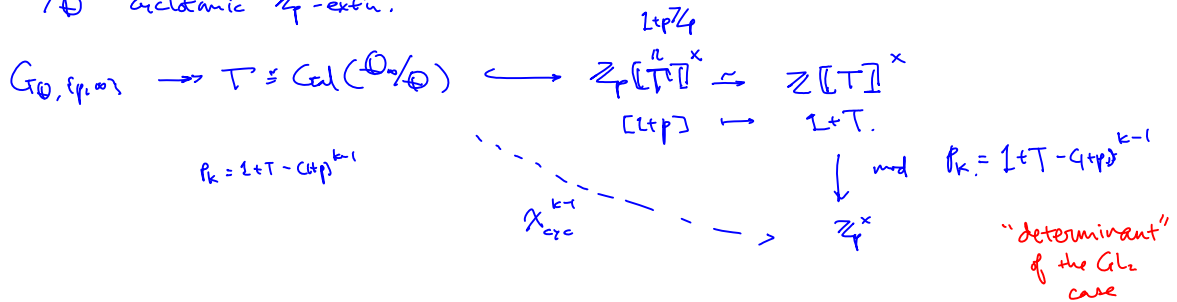
(Put these in families)

GL₂ theory: Observation

$$n \equiv 1 \pmod{p}$$

$\mathbb{Z} \xrightarrow{S \in \mathbb{Z}} n^s$ can be extended p -adic analytically (i.e. congruences between these char's.)

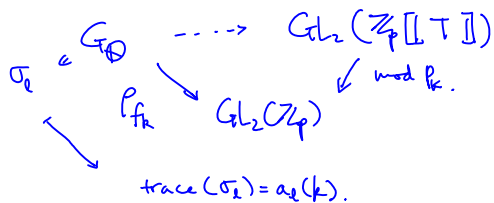
$\mathbb{Q}_\infty/\mathbb{Q}$ cyclotomic \mathbb{Z}_p -ext'n.



Morita: "can glue these cyclotomic char. together"

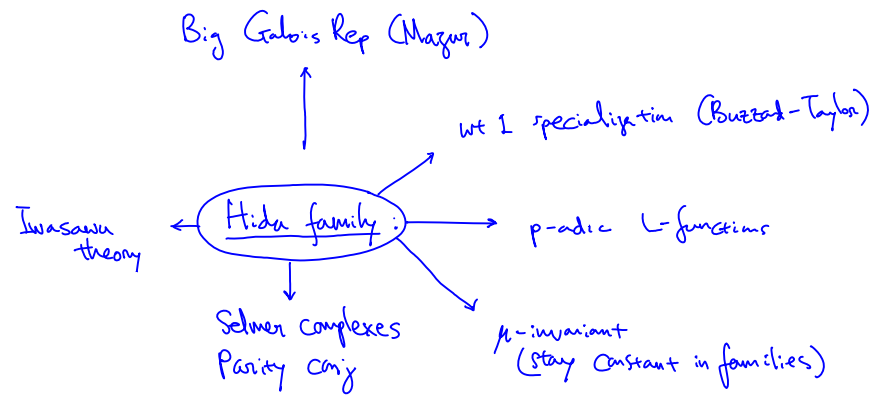
$$GL_2 \quad f_k = \sum_{n \geq 1} a_n(k) q^n$$

want to extend $k \in \mathbb{Z}_{\geq 2} \mapsto a_n(k)$ to a p -adic analytic function of k .



Assumption: Ordinariness, i.e. a_p is a p -adic unit.

Mazur
(copy of Tate Teichmüller
proven by? Serre)



$F = \mathbb{Q}$.

How to construct this family? Constructions: p-adic modular forms

$$\bigcup_{k, \alpha} S_k(\Gamma_1(Np^\alpha); \mathbb{Z}_p)$$

$(N, p) = 1$, the tame level.

(need make a geometric construction to add additional structure.)

→ ① Using (Betti) cohomology of Shimura varieties
(Eichler-Shimura, Natsushikawa (a); Harder)

② Geometric definition $H^0(X, (Np^\alpha)_{\mathbb{Z}_p}, \omega^k)$ and use Igusa tower
↑
automorphic line bundle.

I. Modular Curves ($F = \mathbb{Q}$)

$k=2$ Fix $N \geq 4$

$\alpha \in \mathbb{N}$, $X_1(p^\alpha)$ = modular curve of level $\Gamma_1(Np^\alpha)$.

viewed as inside $\text{End}(H^1)$

Hecke algebra $\mathcal{H}_\alpha = \mathbb{Z}_p[T_n, n \in \mathbb{N}_{>0}] \curvearrowright H^1(X_1(p^\alpha), \mathcal{O}_p/\mathbb{Z}_p) =: M_\alpha$

(advantage:)

$$X_1(p^{\alpha+1}) \xrightarrow{\text{covering}} X_1(p^\alpha) \quad \rightsquigarrow \quad \mathcal{H}_{\alpha+1} \rightarrow \mathcal{H}_\alpha$$

$$M_\alpha \leftarrow M_{\alpha+1}$$

$$M_{\infty} := \varprojlim_{\alpha} M_{\alpha}$$

$$h_{\infty} := \varprojlim_{\alpha} h_{\alpha}$$

* = Pontryagin dual

($\mathbb{Z}_p \leftarrow \mathbb{O}_r/\mathbb{Z}_p$ coeff.)

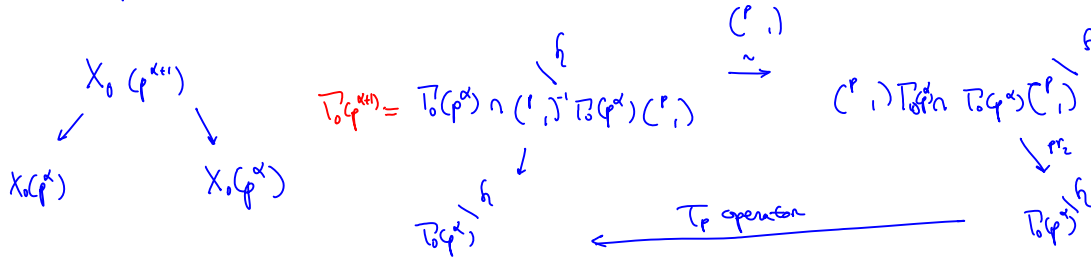
$\begin{array}{c} X_{\alpha} \\ \downarrow \\ X_0(p^{\alpha}) \end{array} \Bigg] \text{ étale of } \mathbb{Z}_p \llbracket (\mathbb{Z}/p^{\alpha})^{\times} \rrbracket \rightarrow h_{\alpha} \subset \text{End}(M_{\alpha}).$

$\Rightarrow M_{\infty}^*$ is a $\mathbb{Z}_p \llbracket \varprojlim_{\alpha} (\mathbb{Z}/p^{\alpha})^{\times} \rrbracket \simeq \mathbb{Z}_p \llbracket \mathbb{Z}_p^{\times} \rrbracket$ -module.

Note the
similarity with
classical Iwasawa
theory!

$M_n^{ord, k} \subset M_n$ the biggest direct \mathbb{Z}_p -factor where T_p is invertible.

$$d_{\infty} = \underbrace{d_{\infty}^{ord}}_{\text{unit}} \times \underbrace{d_{\infty}^{ss}}_{\text{top. nilpotent}}$$



Ordinary assumption $\Rightarrow H_{ord}^2(X_0(p^{k+1})) \simeq H_{ord}^1(X_0(p^k)) \simeq \dots \simeq H_{ord}^1(X_0(p))$

\Rightarrow there is no ordinary families for T_0 !

and

\Rightarrow Families for $T_1 \rightarrow$ deforming the central character $\begin{pmatrix} \chi(c) \\ \chi(c) \end{pmatrix}$

Thm (Hida):

- ① d_{∞}^{ord} is finite and flat over $\mathbb{Z}_p[[\mathbb{Z}_p^{\times}]]$, also over $\Delta = \mathbb{Z}_p[[1+p\mathbb{Z}_p]] \simeq \mathbb{Z}_p[[T]]$.
 $M_n^{ord, k}$ is free of finite type over Δ .

② $\alpha \geq 0$, $P_{\alpha} = (1+T)^{p^{\alpha}} - 1$, then $h_{\alpha}^{ord} / P_{\alpha} M_n^{ord} \simeq h_{\alpha}^{ord}$ $M_n^{ord, k} / P_{\alpha} M_n^{ord} \simeq M_n^{ord, k}$.

(can recover the finite level Hecke algebras and H^i 's.)

III Hilbert modular case.

$F =$ totally real field, $d = [F:\mathbb{Q}]$.

$I = \text{Hom}_{\mathbb{Q}}(F, \bar{\mathbb{Q}})$.



$B =$ quaternion algebra over F . (e.g. $B = M_2(F)$)

ramified only at some infinite places $I_B \subset I$

$G = \text{Res}_{\mathbb{Q}} B^{\times}$ (e.g. $G = \text{GL}_2(F) / \mathbb{Q}$)

$$K = \text{open compact of } G(\mathbb{A}_f) = G_{\mathbb{Z}}(\mathbb{A}_{f,f})$$

$$\prod_v K_v \quad K_v \subset G_{\mathbb{Z}}(G_{F,v})$$

Shimura variety

$$Y_K = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K K_{\infty} \rightarrow (\text{max. compact of } G(\mathbb{R})) \cdot \text{centers}$$

$$Y_K = \text{Shimura variety of dim} = \#(I \backslash I_B) = r$$

↙ class gr?

quasi prof.

smooth if K is small enough

$$K = K_{\mathbb{Z}}(\mathbb{N}) \cap K_{\mathbb{D}}(p^{\infty})$$

Two cases: (I) $t=0$ or $1 \equiv d \pmod{2}$ ↙ Hida varieties ↘ Shimura curve.

(II) $d=r$ $G_{\mathbb{Z}} = G_{\mathbb{Z}}(F)$ (Hilbert modular)

Level at p : $K_0(p^\alpha) = K_1(p^\alpha) = K_n(p^\alpha)$ ← to put into families more content here

$$\begin{pmatrix} * & * \\ & * \end{pmatrix} \quad \begin{pmatrix} * & * \\ & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$$

no families central den. local twist den.

$1 \pm \delta$ -variable families d -variables.

$f = \text{Leopoldt}$

Weight: $w = (w_\tau, \tau \in I; w_0) \quad w_\tau \geq 0 \quad w_\tau \equiv w_0 \pmod{2}$

$$v_\tau = \frac{w_0 - w_\tau}{2}$$

$$M_\alpha^{(w)} = H^r(Y, \mathcal{O}_p^\alpha), \mathcal{L}(w; \mathbb{F}/\mathbb{G})$$

local system: $\otimes_{\mathbb{Z}} \text{Sym}^{w_\tau} \otimes \det^{\frac{w_0 - w_\tau}{2}}$

G
U
B = standard Borel
U
T

$$h_{\alpha}^{(w)} \leftarrow G [T(\mathbb{Z}/p^\alpha) / G_{\mathbb{F}^x}]$$

(image)

$$\Lambda^{n,0} = G [T(\mathbb{Z}/p^\alpha) / G_{\mathbb{F}^x}]$$

$$M_{\alpha}^{(w)} = \varinjlim M_{\alpha}^{(w)}$$

\downarrow
 $M_{\infty}^{n,0}(w)$

$$h_{\infty}^{(w)} = \varprojlim h_{\alpha}^{(w)}$$

\parallel
 $h_{\infty}^{n,0} \times h_{\infty}^{(w)}$

$$T_p^0 = \prod_{\mathbb{F}^x} T_v^0 = T_r \cdot p \text{ some power}$$

(so invertible)

$$\begin{matrix} Y_1(p^\alpha) \\ \downarrow \\ Y_0(p^\alpha) \end{matrix} \quad T(\mathbb{Z}/p^\alpha) / G_{\mathbb{F}^x} \simeq (G_{\mathbb{F}^x}/p^\alpha) / G_{\mathbb{F}^x}$$

$$\begin{pmatrix} wz \\ z \end{pmatrix} \leftarrow (wz)$$

Thm (Independence of w_τ , Hida)

$$h_{\infty}^{n,0}(w) \simeq h_{\infty}^{n,0}(0) = h_{\infty}^{n,0} \quad \text{as } \Lambda\text{-modules}$$

$$M_{\infty}^{n,0}(w) \simeq M_{\infty}^{n,0}(0) = M_{\infty}^{n,0}$$

where the action of $\Lambda^{n,0}$ on w is twisted by

$$\Lambda_{n,0} \rightarrow \Lambda_{n,0}$$

$$[u, z] \mapsto u^v z^{w_0} [u, z]$$

$$\begin{aligned} M_{\infty}^{n,0}(w) &\stackrel{\text{def}}{=} \varinjlim_{\alpha} H_{n,0}^r(Y_n(p^\alpha), \mathcal{L}(w, \mathbb{F}/\mathbb{G})) \\ &= \varinjlim_{\alpha} H_{n,0}^r(Y_1(p^\alpha), \mathcal{L}(w, \mathbb{F}^d/\mathbb{G})) \\ &= \text{" (} \mathbb{F}^d/\mathbb{G}/n \end{aligned}$$

$$\leq M_{100}.$$